ON THE BIFURCATIONS OF STATIONARY MOTIONS OF CONSERVATIVE SYSTEMS WITH TWO CYCLIC COORDINATES

PMM Vol. 31, No. 5, 1967, pp. 841-847
V. I. VOZLINSKII
(Moscow)
(Received April 10, 1967)

This paper is concerned with the stability of the stationary motions of systems with two cyclic coordinates, and is based on the investigation of Rumiantsev [1]. Some properties of bifurcation are established for a particular type of sections of the surface of stationary motions. The particular case of the bifurcation pointed out in Ishlinskii's [2] is investigated.

1. Let us first consider some questions of the theory of bifurcation [3 and 4], which are necessary further on when the theory is generalized to the case in which the potential energy depends on two parameters.

Let the system have k degrees of freedom; $x = \|x_1, \ldots, x_k\|$ is a vector in configuration space R; p_1 and p_2 are real parameters varying on the axes P_1, P_2; $p = \|p_1, p_2\|$ is the corresponding vector of the $P = P_1 P_2$ surface; $\Pi(x, p_1, p_2)$ is the potential energy which is assumed to be analytic.

The equation of equilibrium

$$\operatorname{grad}_x \ \Pi(x, p_1, p_2) = 0 \qquad (1.1)$$

determines in the space RP the surface of equilibria B (see [1]). The smooth surfaces $C_s \subset B$ are called branches of the surface B. The point of bifurcation of the equilibria is defined, as was done in [5], as the point of branching of the solutions of the equilibrium equation. The point (x^0, p^0) of the equilibrium surface is called a point of bifurcation of the equilibria, if in any (no matter how small) neighborhood of this point there exists at least two points (x', p^*), (x'', p^*), belonging to the equilibrium surface and corresponding to one value p^* of the parameter. A necessary (but not sufficient) condition for some point of the surface B to be a point of bifurcation is that the Hessian of the potential energy be equal to zero at that point [4].

The set Γ of the points of bifurcation is called the bifurcation curve. In particular the lines of intersection of the branches of the surface B belong to that curve.

The law of variation of stability for fixed values of p is satisfied on the surface B (see e.g. [4]).

In some cases it is worth considering not the entire surface B, but only some curve L located on it, and which is the section of the surface B by the $(k+1)$-dimensional surface given by the smooth function

$$p_2 = p_2(x, p_1) \qquad (1.2)$$

(or $p_1 = p_1(x, p_2)$; such a notion is useful later on). If (1.2) does not depend on x, the section is said to be cylindrical (in that case (1.2) determines a $(k+1)$-dimensional "cylindrical" surface, the "generatrices" of which are orthogonal complements of the plane P).

Substituting (1.2) into (1.1) we get the projection L^* of the curve L on the subspace

RP_1. By stable (unstable) points of L^* we shall mean projections of stable (unstable) points of the curve L. There will result a certain distribution of the stability on L^*. Thereby it is not known beforehand, whether it corresponds to the laws [3 and 4] (as on the equilibrium curve of systems with one parameter). If such a correspondence exists formally, we shall say that there exists a "Poincaré distribution" or a "regular distribution".

It is easy to see that for a cylindrical section L, the distribution of the equilibrium on L^* is a Poincaré distribution. Indeed, in such a case the points of the curve L^* are stationary points of the function $\Pi^*(x, p_1) = \Pi(x, p_1, p_2(p_1))$ for fixed p_1, i.e. the curve L^* coincides with the equilibrium curve for the potential energy $\Pi^*(x, p_1)$, and the distribution of the equilibrium coincides also with them. This leads to the classical case of one parameter. The section which has several branches on the bifurcation curve is not an exception either, in spite of the fact that on such branches and on corresponding branches of the curve L^*, the Hessian of the potential energy is identically equal to zero. This case is considered in [5].

If L is not a cylindrical section, then the stability distribution on L^* can be not regular (in the accepted sense).

This follows from the following reasoning. If L is not a cylindrical section, then to points $M_1^*, M_2^*, \ldots$ on L^* for a fixed value of p_1, there corresponds points $M_1, M_2, \ldots$ on L for different values of the vector p (only the coordinate p_1 of this vector is fixed). Thus the law of the variation of stability may not be satisfied on $L^* \subset RP_1$, while at the same time it is satisfied (for fixed p) on L in the entire space RP.

Using the cylindrical sections, it is easy to show (taking [5] into consideration) that the boundary of the regions of stability and instability in B belongs to the bifurcation curve Γ.

Note 1.1. If the number of parameters is equal to $m > 2$, then B is an m-dimensional surface and Γ a manifold of dimensions $\leq m - 1$. We may consider sections of different dimension.

2. Let us consider a nongyroscopically coupled conservative system with k positional and two cyclic coordinates.

1°. There exists an analogy [1] between the theory of bifurcation of the stationary motions of such systems and the theory of bifurcation of equilibria. In agreement with that analogy, the notations of Section 1 take the following meaning: x is a k-dimensional vector of the position coordinates; p is a two-dimensional vector of the generalized impulses corresponding to the generalized velocities of the cyclic coordinates (*); B is the surface of stationary motions in the $(k+2)$-dimensional space RP, determined by Eq.

$$\operatorname{grad}_x W(x, p) = 0 \qquad (2.1)$$

where W is Routh's potential (reduced potential energy) [1 and 6]

$$W = \Pi(x) + {}^1\!/_2 \Sigma\, b_{ij}(x)\, p_i p_j = \Pi + {}^1\!/_2 (Bp, p) \qquad (2.2)$$

Here $\Pi(x)$ is the potential energy; $A = \|a_{ij}\|$ $(i =, j = 1, 2)$ is the kinetic energy matrix, corresponding to transformations of the cyclic velocities; $B = \|b_{ij}\| = A^{-1}$ is its

*) These impulses and velocities will be called cyclic for short. Cyclic impulses on the motions of the system are equal to constants, which will be denoted as the impulses by p_1, p_2.

inverse matrix.

We shall also introduce the notation $\omega = \|\omega_1, \omega_2\|$ for the vector of the cyclic coordinates, which varies on the surface Ω $(\omega_1 \in \Omega_1, \ \omega_2 \in \Omega_2)$.

In gyroscopically noncoupled systems, the cyclic impulses and cyclic velocities are connected by the relation [6] $p = A(x)\,\omega \qquad (\omega = B(x)\,p)$ (2.3)

The cyclic coordinates are constant on stationary motions, thus it is possible to talk about the surface B_ω of the stationary motions in the space $R\Omega$; the cyclic velocities ω_1, ω_2 [7] appear as parameters of this surface. However, as is shown in [1], the stability distribution on B_ω, is not a Poincaré distribiution. The change from B to B_ω can be treated as the change of variables (2.3).

Note 2.1 (see [1]). In a gyroscopically coupled system, the surface B is also determined by Eq.(2.1), and the surface B_ω is connected with it by the relation (2.3). But the stability distribution on these surfaces can be differentiated from the distribution found in the reduced system when the gyroscopic forces are not taken into account: on the segments having an even degree of instability, it is possible to have a gyroscopic stabilization.

2°. Let us consider the cylindrical section L_ω of the surface B_ω. Let it be, for the sake of definiteness, the section by the hyperplane (*)

$$\omega_2 = \lambda\,\omega_1 \qquad (\lambda \text{ is a number}) \tag{2.4}$$

The section $L \subset B$ in the space RP corresponds to the section L_ω (according to (2.3)). We shall denote, respectively, by L_ω^* and L^* the projection of the section L_ω on the subspace $R\Omega_1$, and that of the section L on the subspace RP_1. The curves L_ω^* and L^* are connected by the relation

$$p_1 = (a_{11}(x) + \lambda a_{12}(x))\,\omega_1 \tag{2.5}$$

Now, the section L is not cyclindrical, thus the passage from L_ω^* to L^* according to (2.5) leads, in general, from an nonregular distribution (in the accepted sense) to another nonregular one, in spite of the fact that it is the impulse p_1, and not the velocity which is a parameter on L^* (the passage from B_ω to B according to (2.3) leads from a nonregular distribution to a regular one [1]).

Further on we shall consider those laws of the Poincaré distribution which are valid not only on a cyclindrical section of the surface B, but also (under certain conditions), on a cylindrical section of the surface B_ω, and in systems with one cyclic coordinate on the curve B_ω.

3°. Let the transformation (2.3) be nondegenerate for $x = 0$

$$\det A(0) \neq 0 \tag{2.6}$$

For systems with one cyclic coordinate (for them A is a number) the condition (2.6) means that the kinetic energy, with respect to ω, is not identically equal to zero, on the stationary motion $x = 0$; this condition is practically always fulfilled (in the opposite case, the problem degenerates into a static one, as, for instance, the problem of the stability of the trivial stationary "motion" of a mass point suspended on a string).

From [3 and 5] and what has been said above follow the following assertions.

---

*) It is assumed that hyperplane (2.4) is not tangent to the surface of stationary motions.

1) Let in the system with one cyclic coordinate, the curve B_ω contain a trivial branch: the curve $x = 0$, and let Eq. (2.6) be satisfied. Then the following statements are valid for the curve B_ω:

a) the points of stability change on the trivial branch are bifurcation points;

b) the isolated bifurcation points on the trivial branch for $\omega \neq 0$ are either points of stability change, or are located inside the domain of instability.

2) Let the cylindrical section L_ω of the surface B_ω by the hyperplane (2.4) contain the trivial branch $x = 0$, and assume the condition (2.6) is fulfilled. Then the statements (a) and (b) are satisfied on the projections $L_\omega^* \subset R\Omega_1$, of that section.

In fact, it is easy to see that the statements (a) and (b) are satisfied in RP (see [5]). Furthermore, in systems with one cyclic coordinate, when the condition (2.6) is satisfied there is a one to one correspondence between the points of bifurcation of B which are nonlimiting [3 and 5] and the nonlimiting points of bifurcation of B_ω. But on the trivial branch $x = 0$ all the points of bifurcation are nonlimiting. Hence follows the assertion (1). Taking the last foot-note into consideration, it is also easy to prove the assertion(2).

Note 2.2. In the assertion (2) we considered the cylindrical section by the hyperplane (2.4), however, the statement (a) is valid also for arbitrary cylindrical sections. In connection with the statement (b) we shall note that for cylindrical sections other than (2.4), the Routh potential does not reduce, in general, to the form (2.4) of [5]. For such sections the statement (b) is valid in the case in which there are an odd number of branches passing through the points of bifurcation under consideration (under certain conditions [4 and 5] which are usually met in practical problems). Usually, at a point of bifurcation, the branch $x = 0$ intersects with a nontrivial branch.

4°. Let us consider the case of a transformation (2.3) which is degenerate for $x = 0$

$$\det A\ (0) = 0 \tag{2.7}$$

It has a practical interest in systems with two cyclic coordinates when rank $A\ (0) = 1$.

Expressions (2.2), (2.1) are indefinite for $x = 0$. Since $A\ (0)$ is a symmetric matrix of second order, the eigenvector g of this matrix is orthogonal to the eigenvector of the adjoint matrix $A^s\ (0)$. Since furthermore, the matrix $A\ (0)$ is sign-definite, the quadratic form $(A^s\ (0)p, p)$ is equal to zero, if and only if, $p \in g$. Thus, the quadratic form

$$(B(x)\,p,\ p) = \frac{(A^s(x)\,p,\ p)}{\det A\,(x)}$$

which enters (2.2) can have a finite limit for $x \to 0$ only when $p \in g$ (see e.g. in [6]). The same applies to the expression $\mathrm{grad}_x W(x, p)$. It follows that for the conditions (2.7) either there are no stationary motions $x = 0$, $p \neq 0$, or the surface B contains the line g and only it for $x = 0$. The entire plane $x = 0$, corresponds to that line, in the domain $R\Omega$. Hence it is clear that in the case (2.7) the statement (b) of the Subsection 3° may be invalid on the cylindrical sections L_ω of the surface B_ω (but it obviously remains also valid in the case (2.7) on the cylindrical sections of the surface B).

5°. As an illustration of Subsections 2° to 4°, we shall investigate the surface B of Fig. 1, which corresponds approximately to the surface of the stationary motions of an elongated body with a fixed point in the case of Lagrange. Here θ is the nutation angle $0 \le \theta \le \frac{1}{2}\pi$ ($\theta = 0$ if the center of gravity is situated below the point of support on the vertical line going through that point); p_1 and p_2 are cyclic impulses corresponding to the velocities $\omega_1 = \psi^\cdot$ and $\omega_2 = \varphi^\cdot$ of the precession and natural rotation respectively (the part of the surface corresponding to $p_2 > p_1$ is not shown). The linear surface E

(cross-hatched in Fig. 1) is determined by the condition $\omega_2 = 0$; L represents the corresponding section. On Fig. 2 is shown the projection L^* of this section on the plane θp_1. The little circles mean that the curves L and L^* are stable (*). The fact that Fig. 2 seems to contradict the law of stability variation is explained by noticing that the section L is not cylindrical. As can be seen in Fig. 1, there is no such contradiction on cylindrical sections in the space $\theta p_1 p_2$. In the space $\theta \omega_1 \omega_2$, the statement (b) of the Subsection 3° obviously does not hold on the cylindrical section by the plane $\omega_2 = 0$. This is explained by the degenerate nature of the matrix A for $\theta = 0$ (cf. relation (22) in [6]).

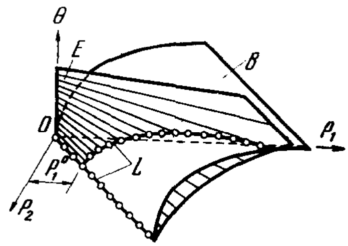

Fig. 1

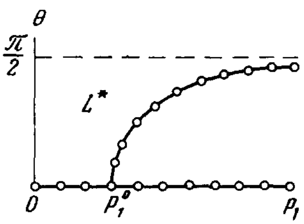

Fig. 2

3. Let us consider system of the paper [2]: an elongated body with axial symmetry, suspended on an absolutely flexible inertialess string, the top end of which rotates freely in a vertical bearing, and its low end (the point O) is tied to the axis of symmetry of the body above its center of gravity (point C). We shall take the point O_1 in which the string comes out of the bearing as the origin of a fixed orthogonal system of coordinates $O_1\xi\eta\zeta$ which has its axis ζ directed vertically upwards. The point C is taken as the origin of the two systems $C\xi'\eta'\zeta'$ and $Cxyz$. The axes of the first are directed parallel to the axes ξ, η, ζ ; the axes of the second (rigidly fixed to the body) are considered as the principal central axes of inertia of the body whereby the axis z is directed along the axis of symmetry towards the point O.

The given mechanical system has five degrees of freedom. Let us consider the two systems of generalized coordinates

$$1)\ \alpha, \beta, \theta, \psi, \varphi; \quad 2)\ \alpha, \beta, \theta, \gamma, \varphi \tag{3.1}$$

Here θ, ψ, φ are the angles of nutation of the precession and the proper rotation of the body in the system $C\xi'\eta'\zeta'$, $Cxyz$; α and β are the angles of nutation and precession of the string as a rigid body in the system $O_1\xi\eta\zeta$ (the axis of the string is assumed to be directed from O to O_1); $\gamma = \beta - \psi$. It is assumed that $0 \leq \alpha \leq \frac{1}{2}\pi$; $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$. We note that the angles ψ and β correspond exactly (including their signs) to the angles between the axis η and the respective projections on the horizontal plane of the axis z and the axis of the string [8]. It is easy to verify that in the system of coordinates (3.1_1) there is one cyclic coordinate φ : in the system of coordinates (3.1_2) there are two cyclic coordinates: β and φ, i.e. the maximum possible number. Therefore we shall use the system of coordinates (3.1_2) in the future. The mechanical system under consideration is gyroscopically coupled.

In [2 and 8] (with the notation introduced above) the velocity $\beta^{\cdot}$ of the cyclic coordinate β is chosen as a parameter, and the investigation is concerned with the stationary motions on which the velocity $\varphi^{\cdot}$ of the other cyclic coordinate is equal to zero. Thus one considers the cylindrical section of the surface B_ω of the stationary motions, given

*) Compare with example from [1 and 7] which is obtained from the case under consideration by taking $\omega_2 = 0$.

by the condition $\varphi^{\cdot} = 0$. The curve (6) of [2] represents the projection of this section on the subspace $\alpha\theta\beta^{\cdot}$. It has a peculiarity: the trivial branch does not lose its stability in the points of bifurcation (i.e. the statement (b) of Section 2 does not hold on it). This peculiarity is retained also when the parameter is not the velocity $\beta^{\cdot}$ but the corresponding impulse p_β. It is shown further on, that this peculiarity is obtained because the transformation (2.3) is degenerate on the trivial motion; the fact that it is also retained for the parameter p_β can be explained by observing that the condition $\varphi^{\cdot} = 0$ yields in the RP space a noncylindrical section.

Let us introduce the following nondimensional parameters and functions:

$$m^2 = \frac{Ml^2}{A}, \quad n^2 = \frac{Ma^2}{A}, \quad \nu = \frac{C}{A}, \quad \omega_1 = \frac{A\beta^{\cdot}}{\mu}$$

$$\omega_2 = \frac{A\varphi^{\cdot}}{\mu}, \quad p_1 = \frac{p_\beta}{\mu}, \quad p_2 = \frac{p_\varphi}{\mu} \left(\mu = \sqrt{Ag\sqrt{AM}}\right)$$

$$\Phi(\alpha, \theta, \gamma) = m^2 \sin^2\alpha + n^2 \sin^2\theta + 2mn \sin\alpha \sin\theta \cos\gamma$$

Here ℓ represents the length of the string (between O_1 and O) ; a is the distance between the points O and C, ($\ell > a$) ; M is the mass of the body ; g is the acceleration due to gravity ; A, C ($A > C$) are the principal cenral moments of inertia of the body, the equatorial and the axial, respectively ; p_β and p_φ are the cyclic impulses ; ω_1, ω_2, p_1 and p_2 are nondimensional cyclic velocities and impulses. Further on, all expressions are written with accuracy up to the normalizing factors.

The matrix of the part of the kinetic energy which appears as a quadratic form of the cyclic velocities, its inverse matrix and Routh's potential have, respectively, the following form:

$$A \| a_{ij} \| = \begin{Vmatrix} \nu\cos^2\theta + \sin^2\theta + \Phi & \nu\cos\theta \\ \nu\cos\theta & \nu \end{Vmatrix} \tag{3.2}$$

$$B = \| b_{ij} \| = \frac{1}{\nu(\sin^2\theta + \Phi)} \begin{Vmatrix} \nu & -\nu\cos\theta \\ -\nu\cos\theta & \nu\cos^2\theta + \sin^2\theta + \Phi \end{Vmatrix}$$

$$W = -(m\cos\alpha + n\cos\theta) + \frac{(p_1 - p_2\cos\theta)^2}{2(\sin^2\theta + \Phi)} + \frac{p_2^2}{2\nu} \tag{3.3}$$

The equations of the stationary motions which determine the surface B in the space RP of the variables $\alpha, \theta, \gamma : p_1, p_2$ are

$$\frac{\partial W}{\partial\alpha} = m\sin\alpha - \frac{\Phi_\alpha (p_1 - p_2\cos\theta)^2}{2(\sin^2\theta + \Phi)^2} = 0 \tag{3.4}$$

$$\frac{\partial W}{\partial\theta} = n\sin\theta + \frac{\sin\theta(p_1 - p_2\cos\theta)}{(\sin^2\theta + \Phi)^2}\left[p_2 - p_1\cos\theta + p_2\Phi - \frac{(p_1 - p_2\cos\theta)\Phi_\theta}{2\sin\theta}\right] = 0 \tag{3.5}$$

$$\frac{\partial W}{\partial\gamma} = -\frac{(p_1 - p_2\cos\theta)^2\Phi_\gamma}{2(\sin^2\theta + \Phi)^2} = 0 \tag{3.6}$$

Here Φ_α, Φ_θ and Φ_γ are the partial derivatives of Φ with respect to α, θ and γ.

It follows from (3.4), (3.5) that there are no stationary motions on which $\alpha = 0$, $\theta \neq 0$, $\gamma \neq 0$ nor stationary motions on which $\theta = 0$, $\alpha \neq 0$, $\gamma \neq 0$; from (3.6) it follows that for $\alpha \neq 0$, $\theta \neq 0$ the stationary motions surface lies in the subspace $\gamma = 0$ (this was pointed out in [8]). When $\alpha = \theta = 0$ there is a practical reason to search the stationary motions for $\gamma = 0$, which is done below.

For $\alpha = \theta = 0$, the matrix (3.2) is degenerate, and Expressions (3.3) to (3.6) are indeterminate. These indeterminate forms can be evaluated on the eigenvector $p_1 = p_2$ of the matrix (3.2). To wit, taking into consideration that $\Phi(\alpha, \theta, \gamma)$ is a sign definite function for any α, θ, γ, it is easy to verify that the limits of Expressions (3.4) to (3.6) and the second factored Expression of (3.3) are equal to zero for $p_1 = p_2$, $\alpha \to 0$, $\theta \to 0$ (γ is arbitrary). Consequently, the surface B contains the branch $p_1 = p_2$ for $\alpha = \theta = 0$. It can be verified that for $p_1 = p_2$, $\alpha \neq 0$, $\theta \neq 0$ the system of Eqs. (3.4) to (3.6) is not compatiable, i.e. the cylindrical section by the hyperplane $p_1 = p_2$ represents (in the subspace $\gamma = 0$) a single line $\alpha = \theta = 0$, $p_1 = p_2$. This line is entirely stable, but stability is being considered with respect to the part of the variables α, θ, $\alpha^\cdot$, $\theta^\cdot$, p_1 and p_2. In fact, taking into consideration that the gyroscopic forces [1] cannot upset stability, there follows from [9] that for such a stability of the stationary motion $\alpha = \theta = \gamma = 0$, $p_1 = p_2$ it is sufficient that the increment of Routh potential be a positive definite function with respect to the part of the variables α, θ (i.e. that it be sign definite and equal to zero if and only if $\alpha = \theta = 0$ [9]). It is easy to show that Routh's potential (3.3) has that property.

In view of the degenerateness of the matrix (3.2) for $\alpha = \theta = 0$, the entire plane Ω which is also stable corresponds to the investigated straight line in the domain $R\Omega$. Therefore the trivial branch of any cylindrical section in $R\Omega$ is also stable, independently of the presence on it of points of bifurcation. This refers particularly to the cylindrical section by the hyperplane $\omega_2 = 0$ considered in [2]. We can get the projection L_ω^* of this section on the subspace $\alpha\theta\omega_1$, by substituting (2.3) into (3.4) and (3.5) for $\omega_2 = 0$, $\gamma = 0$ (it coincides with the curve (6) from [2] up to the notations). The projection L^* on the subspace $\alpha\theta p_1$ can be obtained by substituting $p_1 = a_{11}(x)\omega_1$ (see (2.5)) into the equation of the curve L_ω^*. Obviously L^* has the same property as L_ω^*, whereby the condition $\omega_2 = 0$ prescribes in RP a noncylindrical section L. The principal feature of this phenomenon can be seen on Fig. 1 and 2 which are concerned with the simpler example (see Section 2).

BIBLIOGRAPHY

1. Rumiantsev, V. V., On the stability of steady motions. PMM Vol. 30, N°. 5, 1966.
2. Ishlinskii, A. Iu., Example of a bifurcation not leading to unstable forms of the stationary motion. Dokl. Akad. Nauk SSSR, Vol. 117, No. 1, 1957.
3. Poincaré, H., Sur l'équilibre d'une masse fluide animée d'un mouvement de rotation. Acta Math., Vol. 7, 1885.
4. Chetaev, N. G., Stability of Motion. Works in Analytical Mechanics. M., Izd. Akad. Nauk SSSR, 1962.
5. Vozlinskii, V. I., On the relations between the bifurcation of the equilibria of conservative systems and the stability distribution on the equilibria curve. PMM Vol. 31, No. 2, 1967.
6. Gantmakher, F. R., Lectures in Analytical Mechanics. M., Fizmatgiz, 1960.
7. Neimark, Iu. I. and Fufaev, N. A., On stability of stationary motions of holonomic and nonholonomic systems. PMM Vol. 30, No. 2, 1966.
8. Temchenko, M. E., On the stability of one of the positions of dynamical equilibrium of a mechnical system. Dokl. Akad. Nauk SSSR, Vol. 117, No. 1, 1957.
9. Rumiantsev, V. V., On the stability of motion with respect to some of the variables. Vest. Mosk. Univ., No. 4, 1957.

Translated by A. V.